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The Spectrum and Confinement for the Bethe-Salpeter Equation.

A.Yu. Umnikov and F.C. Khanna

Theoretical Physics Institute, Physics Department, University of Alberta,
Edmonton, Alberta T6G 2J1,
and TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C. V6T 2A3, Canada

Abstract

The problem of calculating of the mass spectrum of the two-body Bethe-Salpeter equation is studied with no reduction to the three-dimensional ("quasipotential") equation. The method to find the ground state and excited states for a channel with any quantum numbers is presented. The problem of the confining interaction for the Bethe-Salpeter equation is discussed from the point of view of formal properties of the bound state spectrum, but with only an inspiration from the QCD. We study the kernel that is non-vanishing at large Euclidian intervals, $R_E \to \infty$, which is constructed as a special limiting case of a sum of the covariant one-boson-exchange kernels. In the coordinate space this kernel is just a positive constant and corresponds to the kernel $\propto \delta(k_E)$ in the momentum space. When the usual attractive interaction is added, it is found that this kernel is similar in its effect to the non-relativistic potential in coordinate space, $V(r)$, with $V(r \to \infty) \to V_\infty$. The positive real constant V_∞ gives the scale that define the limit of bound state spectrum compared to the sum of the constituent masses, $M < 2m + V_\infty$. At the same time the self-energy corrections remove the singularities from the propagators of the constituents, i.e. constituents do not propagate as free particles. Combination of these features of the solutions allows an interpretation of this type of interaction as a confining one. The illustrative analytical and numerical calculation are presented for a model of massive scalar particles with scalar interaction, i.e. the "massive Wick model".

1 Introduction

The Bethe-Salpeter (BS) equation [1, 2] provides a consistent covariant description for bound state of two fermions (or fermion-antifermion). In particular, a phenomenological success has been achieved in the case of the deuteron describing ground state properties and various reactions [3, 4]. It has been established that BS formalism reproduces the bulk of the non-relativistic approaches at the nucleon momenta small comparing to the nucleon mass, $|\mathbf{p}| < m$, but displays a relativistic effects at larger momenta $|\mathbf{p}| \geq m$ and in some of the spin-dependent observables. However, the relativistic effects are not too large, since the deuteron is essentially a non-relativistic system.

Another topical subject studied within the BS formalism is the bound state of the quark-antiquark systems, mesons ($q\bar{q}$). Compared to the deuteron case the $q\bar{q}$-bound states are governed by a more complex dynamics and, correspondly, physics here is richer. Usual way to proceed here is to reduce the BS equation to an approximate equation of the quasi-potential form and then to use approaches and methods similar to those which are used in the non-relativistic Schrödinger equation.

A number of the specific questions of principle and technical problems must be resolved in order to reach the same state-of-art as it is in the case of the deuteron:

1. First, we need a rigorous method to obtain the bound state spectrum and the corresponding amplitudes for the BS equation. Today we do not find in the literature a discussion of this problem without making approximations.

2. Second, the fundamental problem here is a construction of an effective theory, which would provide the BS equation with confinement. Moreover, it is not clear how the confinement should manifest itself in this formalism.

3. Third problem, we would like to mention here, is the "problem of light quarks (u and d)". For the formalism ofthe BS equation we understand this problem as a description of the dynamical generation of the masses and creating the bound states with a mass, which is much larger then sum of the "bare" masses

of the constituents. Apparently, physics of light quarks is essentially different of physics of heavy quarks. We will not discuss the special problems of light quarks physics in this paper.

All these problems were discussed starting the pioneering work of Llewellyn-Smith [5]. There are, of course, methods to study the spectrum based on the reduction of the two-dimensional integral equations to an *infinite* system of one dimensional equations by means of the hyperspherical harmonics and, then cutting off the system at some lowest functions as an approximation [5, 6, 7]. This method is valid for the tightly bound states, with mass vanishing compared to the sum of the constituent masses, in which case the BS equation has an exact $\mathcal{O}(4)$-symmetry. This was sufficient for early studies, where the mechanism of the confinement was modeled as the binding of the heavy quarks to the relatively light bound states. However, it has become clear that even for heavy quarks the mass spectrum of the mesons lies near and even above the sum of the constituent masses. Therefore other methods must be employed to at the spectrum with such characteristics.

Alternative approaches to the study of the spectrum of the relativistic bound state is based on the reduction of the BS equation to one of the possible quasi-potential equations and then utilizing the Schrödinger-like potentials, $V \sim \alpha r$, in the coordinate space [8] or its generalization in the momentum space [9, 10, 11]. In spite of the obvious advantages of these approaches, such as clear physical interpretation of interactions and solutions with relationship to the Schrödinger equation approach [12], still there is interest to study the full BS equation, which gives direct connection to the covariant field theory.

As to the underlying effective field theory for the relativistic equations of the bound states, now we can only imagine about the structure of such a theory and its connection to the fundamental theory, QCD. The approach based on the integral equations of the field theory [13], the set of Schwinger-Dyson equations for propagators, the Ward-Takahashi relations (Slavnov-Taylor relations) and the Bethe-Salpeter equation is very promissing. The model investigations presented here al-

low for a study of the properties of the analytical structure of the quark propagator [13, 14], the mechanisms of the chiral symmetry breaking and the dynamical generation of the masses [9] and the connections with the bound state problem [15] and observables of reactions [16].

In the present paper we (i) consider the problem of the spectrum of the two-body BS equation with no reduction to the quasipotential equation or other approximations and (ii) attempt to construct a model for the confinement, based on the formal properties of the kernels, but with no direct connection with QCD. We present, in Section 2, a method to find the ground states and excited states in the channel with any quantum numbers. The illustrative numerical calculations, aimed to show capabilities of the method are presented in Section 2.3. We consider in Section 3 the formal properties of the kernel in the form of a sum of the ladder kernels. We derive conditions when this kernel displays properties, similar to the non-relativistic constant potential in the coordinate space. The implementation of this interaction to the single particle propagator leads to the disappearance of the singularities in the propagator. Using the developed method to study the bound state spectrum we analyze the BS equation with this kernel and self-energy corrections in Section 3.3. Combination of properties of the solutions in this case allows for an interpretation of the interaction of this type as the confining interaction.

Nota bene. In this paper we often use the name "quark" for the constituents and the "meson" for the bound system. This is just a matter of convenience and our concrete calculations are not based on the QCD and do not pretend to a phenomenological application. Moreover, the illustrative analytical and numerical calculations are presented for a model with massive scalar particles with scalar interactions, the "massive Wick model". However, the methods can be applied directly to the BS equation for two spinor fields interacting via a phenomenological $q\bar{q}$-interaction. The spectrum of the spinor-spinor BS equation has been studied in ref. [17] with no confining interaction, and the application of the model with the confinement to the meson spectrum will be done elsewhere.

2 The bound state spectrum of the Bethe-Salpeter equation

2.1 The scalar Bethe-Salpeter equation

The BS equation for bound state of two equal mass scalar fields, "quarks", interacting via scalar exchanges has a form:

$$\psi(p,P) = D(p_1,p_2)^{-1} \int \frac{d^4p'}{(2\pi)^4} \mathcal{K}(p,p',P)\psi(p',P), \tag{1}$$

where $\psi(p,P)$ is the BS amplitude for the bound state with momentum P, $\mathcal{K}(p,p',P)$ is a kernel of BS equation and $D(p_1,p_2)^{-1}$ is the two-body propagator. Generally speaking, the kernel $\mathcal{K}$ is a sum of the all renormalized irreducible graphs presenting the Green's function with two incoming and two outcoming constituent fields and the propagator D^{-1} is product of the renormalized full single-particle propagators. In practice, however, both of them are usually taken in the lowest order in the coupling constant. It corresponds to the ladder approximation for the kernel with free particle propagators. In the simplest case of an exchange field of mass μ and coupling constant g, the ladder approximation for the kernel has an explicit form:

$$\mathcal{K}_{ladder}(\mu,p,p',P) = \mathcal{K}_{ladder}(\mu,p,p') \equiv \frac{ig^2}{(p-p')^2-\mu^2}, \tag{2}$$

The two-body propagator then reads

$$D(p_1,p_2) \equiv d(p_1)\cdot d(p_2) = \left(p_1^2-m^2\right)\cdot\left(p_2^2-m^2\right), \tag{3}$$

where $d(p)^{-1}$ is free one-particle propagator, m is the mass of the constituents and $p_{1,2} = \frac{1}{2}P \pm p$ are the constituent momenta.

Phenomenological applications of the equation (1) (or more interesting spinor-spinor equation) requires a more general form of the kernel than eq. (2). More degrees of freedom are provided by introducing a sum of the exchanges with different parameters and different Lorents structure, e.g. vector or pseudoscalar exchanges. In the presence of the several (effective) exchanges with different quantum numbers

and in higher orders kernel may contains both attractive and repulsive terms. Here we consider the kernel of the form:

$$\mathcal{K}_G(p,p') = \sum_{j=1}^{N} \epsilon_j \frac{ig_j^2}{(p-p')^2 - \mu_j^2}, \tag{4}$$

where index j enumerates different "exchange" terms, distinguished by the mass parameter, μ_j, and strength, g_j. Factor $\epsilon_j = \pm 1$ defines attractive ($\epsilon_i = +1$) or repulsive ($\epsilon_j = -1$) type of the correspondent term. Being interpreted as contribution of the lowest order diagram, the terms with negative value of ϵ_j require either antihermitian term in the interaction hamiltonian (lagrangian) or fields with additional quantum numbers. However, we consider the entire sum in the eq. (4) as a convenient parametrization of the unknown full kernel of the BS equation. The convenience of such a form of the kernel is that kernel is explicitly covariant and contains only "field theoretical" elements, the free particle propagators. Accepting this way of action, we assume that parameters of the kernel should be fixed to describe the experimentally known spectrum of the system.

The free propagator $D(p_1,p_2)^{-1}$ of the form (3) and the kernel in the ladder approximation, (2) or (4), usually provide the basis for an investigation of the BS equation. The parameters for a phenomenological adjustment of the theoretical framework are the coupling constants, g_j, and exchange masses, μ_j. The parameter m in the propagator is referred to as "physical mass" of the constituent and it is supposed to include effectively the self-energy corrections. In the case of the observable particles, e.g. nucleons, m is the observable mass, otherwise, e.g. quarks, it has rather ambiguous model-dependent value.

In order to study the spectrum of the BS equation we, first, fix the frame of calculation as the rest frame of the system, where $P = (M, \mathbf{0})$ and M is the mass of the system, "meson". Then, we perform the well-known Wick rotation [18], which may later cause difficulties in the calculation of observables of some reactions, but does not affect the spectrum of the equation and simplifies the numerical analysis.

Under the Wick rotation the BS equation keeps the form (1) with the ladder kernel

$$\mathcal{K}_{ladder}(\mu, p, p') = \frac{g^2}{(p-p')^2_E + \mu^2}, \quad (p-p')^2_E = (p_0 - p'_0)^2 + (\mathbf{p} - \mathbf{p}')^2, \tag{5}$$

and propagator

$$D(p, M) = \left[(p_0^2 + m^2 + \mathbf{p}^2 - M^2/4)^2 + M^2 p_0^2\right]. \tag{6}$$

We use the previous notation for the "rotated" functions ψ, $\mathcal{K}$ and D, and for the "Euclidean" momenta, p and p'.

Next, we perform a partial wave decomposition of the equation [19]:

$$\frac{1}{(p-p')^2_E + \mu^2} = \frac{2\pi}{\mid \mathbf{p} \mid \cdot \mid \mathbf{p}' \mid} \sum_{L=0}^{\infty} \sum_{\lambda=-L}^{L} Q_L(\beta) Y_{L\lambda}(\theta_p, \varphi_p) Y^*_{L\lambda}(\theta'_p, \varphi'_p), \tag{7}$$

$$\psi(p, M) = \frac{1}{\mid \mathbf{p} \mid} \sum_{L=0}^{\infty} \sum_{\lambda=-L}^{L} \psi_L(p_0, \mid \mathbf{p} \mid, M) Y_{L\lambda}(\theta_p, \varphi_p), \tag{8}$$

where we have already taken into account that ψ_L are independent on the projection of angular momentum, λ; the dimensionless parameter β defined by the expression

$$\beta = \frac{\mu^2 + \mathbf{p^2} + \mathbf{p'^2} + (p_0 - p'_0)^2}{2 \mid \mathbf{p} \mid \cdot \mid \mathbf{p}' \mid},$$

and Q_l are the Legendre functions of the second kind. For $l = 0, 1$ they are:

$$Q_0(y) = \frac{1}{2} \mathsf{ln}\left(\frac{y+1}{y-1}\right), \qquad Q_1(y) = \frac{y}{2} \mathsf{ln}\left(\frac{y+1}{y-1}\right) - 1. \tag{9}$$

Substituting (7) and (8) into eq. (1) and performing the angular integration, we arrive at a set of the independent equations for the partial amplitudes ψ_L, corresponding to the states with the angular momentum L:

$$\psi_L(p_0, |\mathbf{p}|, M) = D(p, M)^{-1} \int \frac{dp'_0 d|\mathbf{p}'|}{(2\pi)^3} \mathcal{K}^L(p_0, |\mathbf{p}|, p'_0, |\mathbf{p}'|) \psi_L(p'_0, |\mathbf{p}'|, M), \tag{10}$$

$$\mathcal{K}^L(p_0, |\mathbf{p}|, p'_0, |\mathbf{p}'|) = g^2 Q_L(\beta). \tag{11}$$

The amplitude of pure quantum state with defined angular momentum L and its projection λ is then given by

$$\psi(p, M, L, \lambda) = \psi_L(p'_0, |\mathbf{p}'|, M) Y_{L\lambda}(\theta_p, \phi_p). \tag{12}$$

The two-dimensional integral equations (10) are an exact projection of the initial equation (1) with (2) and (3). In the case of the generalized kernel (4), there will be a sum over different terms on the right hand side of eq. (10):

$$\mathcal{K}_G^L(p_0, |\mathbf{p}|, p_0', |\mathbf{p}'|) = \sum_{j=1}^{N} \epsilon_j g_j^2 Q_L(\beta_j). \tag{13}$$

To obtain the spectrum of the BS equation we need to solve the eigenvalue problem for the bound state mass M at fixed set of parameters of the kernel. =It is to be noted that the BS equation is not linear in the mass M. On the other hand it is linear in the coupling constants squared, g^2. Therefore we consider an equation of the form:

$$\psi_L(p_0, |\mathbf{p}|, M) = \lambda \cdot D(p, M)^{-1} \int \frac{dp_0' d|\mathbf{p}'|}{(2\pi)^3} \mathcal{K}_G^L(p_0, |\mathbf{p}|, p_0', |\mathbf{p}'|) \psi_L(p_0', |\mathbf{p}'|, M), \tag{14}$$

or symbolically

$$\psi = \lambda \cdot \hat{\mathcal{K}} \psi, \tag{15}$$

where we skip indicies L for shortness.

Solving the linear eigenvalue problem for λ at fixed parameters of the kernel and various values of the bound state mass, we can map $\lambda(M)$. Then by inverting the mapping as $M(\lambda)$, we will find solutions of the eigenvalue problem for M with the kernel $\lambda \cdot \hat{\mathcal{K}}$, where factor λ is trivially absorbed by redefining the coupling constants.

Traditional methods to find solutions to the linear eigenvalue problem corresponding to the integral equation (15) are based on the ideas of the classical Fredholm theory. The basic idea here is to substitute integration by a summation and then deal with a sufficiently large system of linear algebraic equations. The problem is reduced to finding the eigenvalues of the matrix corresponding to the integral operator $\hat{\mathcal{K}}$. However, in the case of the covariant BS equation such a program of action meets extra difficulties in view of the double integration, which leads necessarily

to very large matrices, say $\sim 10^4 \times 10^4$ or even larger in the case of spinor-spinor equation (with additional factor of 64 if the parity is conserved by interactions and factor 256 if it does not).

We use an alternate method based on the iteration of the BS equation in the form (10). This method in the essence is similar to the Malfliet-Tjon method [20] employed to solve the integral Schrödinger equation in the nuclear physics (see also discussion in ref. [21]).

2.2 The iteration method

The iteration of any trial function, $\Phi^{(0)}$, with eq. (10), is understood as obtaining other function $\Phi^{(n)}$ using the algorithm (we use the symbolic notations as in eq. (15)):

$$\Phi^{(i+1)} = \lambda \cdot \hat{\mathcal{K}} \Phi^{(i)}, \tag{16}$$

$$\Phi^{(n)} = \left[\lambda \cdot \hat{\mathcal{K}}\right]^n \Phi^{(0)}, \tag{17}$$

In order to organize the iteration (16)-(17) on the computer, we need an "integrator", corresponding to the operator $\hat{\mathcal{K}}$. This means that we have to perform the computer calculation of the double integral on the r.h.s. of the equation with a defined kernel and any trial function, $\Phi^{(0)}$, (we assume good enough behavior of the function, such as absence of singularities and vanishing at large arguments, $|\mathbf{p}| \to \infty$ and $p_0 \to \pm\infty$). In our particular calculations this integrator is organized as a two dimensional Gauss integration with suitable mapping of variables. Next, we would like to know what happens with equation after sufficiently large number, n, of iterations.

Let us assume that solutions, ψ_α, of the equation (15) corresponding to the eigenvalues λ_α belong to the complete system of functions[22]. Therefore, the trial function can be expanded as

$$\Phi^{(0)} = \sum_{\alpha=0}^{\infty} A_\alpha \psi_\alpha. \tag{18}$$

Thus, the result of iteration (16)-(17) is:

$$\Phi^{(n)} = \sum_{\alpha=0}^{\infty} A_\alpha \left[\frac{\lambda}{\lambda_\alpha}\right]^n \psi_\alpha. \tag{19}$$

From last equation it is obvious that at sufficiently large $n = N$ all terms with $\alpha > 0$ are small compared to the ground state term, $\alpha = 0$:

$$\lim_{n\to N} \Phi^{(n)} \equiv \Phi^{(N)} = C \cdot \psi_0 + \mathcal{O}\left(\left[\frac{\lambda_0}{\lambda_1}\right]^N\right). \tag{20}$$

Therefore N to be chosen to make the last term on the r.h.s. of (20) to be negligibly small. Then, comparing $\Phi^{(N)}$ and $\Phi^{(N+1)}$, we find the ground state eigenvalue, λ_0:

$$\lambda_0 = \frac{\Phi^{(N)}}{\Phi^{(N+1)}}. \tag{21}$$

This recipe to find the ground state eigenfunction and eigenvalue works nicely numerically and, of course, does not depend on the choice of the initial trial function.

Formula (18) also provides us with the possibility to find higher levels on λ. Indeed, taking the combination of iterating functions:

$$\Phi^{(i+1)} - \left[\frac{\lambda}{\lambda_0}\right] \cdot \Phi^{(i)} = \sum_{\alpha=1}^{\infty} \left(1 - \frac{\lambda_\alpha}{\lambda_0}\right) A_\alpha \left[\frac{\lambda}{\lambda_\alpha}\right]^{(i+1)} \psi_\alpha, \tag{22}$$

i.e. new trial function which does not contain admixture of the ground state! It is easy to see that iterations of this function give the eigenfunction ψ_1 of the first excited state, λ_1, similarly to the procedure for ground state. The same procedure, in principle, can be organized for any desired level on λ. Thus, the problem is solved. The only limitation is, of course, accuracy of the numeric calculations. Calculation of the high λ require a precise calculation of all levels below, which leads to substantial computer time. No special numerical problems were find in calculating the three lowest levels for eq. (15). The numerical results are presented in the next section.

2.3 Numerical results for iteration method

To study numerically the capability of the method to solve the integral BS equation we consider eq. (10) with the model kernel of the form (13). The parameters of the model are presented in Table I. We refer to the constituent fields as "quarks" and their mass, m, is chosen to be similar to the mass of the c-quark. The parameters of the kernel are chosen to provide the typical density of the levels of the $c\bar{c}$-bound state, the charmonium, not too far from the limit of the spectrum $M_{lim} = 2m$.

coupling constants $g_j^2/(4\pi), \quad GeV^2$	ϵ_j	mass μ_j, GeV
37800.0	+1	0.10
37800.0	-1	0.11
45.0	-1	0.95
45.0	+1	1.425
$m = 1.5$ GeV,		

Table. I. The parameters of model ("scalar charmonium").

Results of a calculation of the spectrum, $\lambda(M)$, are presented in Fig. 1 for the three lowest levels in the channels with $L = 0$ (S-states) and $L = 1$ (P-states). The physical spectrum $M(\lambda)$ can be obtained crossing the plot by the line $\lambda = \lambda_{phys}$. For instance, as is shown in Fig. 1 at $\lambda_{phys} = 0.13$ the masses of the lowest states are (with accuracy $\sim 0.5\%$):

$$M(1S) = 1.265 \quad \text{GeV}; \quad M(2S) = 1.939 \quad \text{GeV}; \quad M(3S) = 2.251 \quad \text{GeV}; \quad (23)$$

$$M(1P) = 1.751 \quad \text{GeV}; \quad M(2P) = 2.146 \quad \text{GeV}; \quad M(3P) = 2.385 \quad \text{GeV}. \quad (24)$$

For illustartion, we pesent the amplitudes ψ_0, corresponding the spectrum (23) in Fig. 2. These amplitudes are shown as a functions of the spatial momentum $|\mathbf{p}|$ and at $p_0 = 0$. We see that the type of radial excitations is similar to the one for the Schrödinger equation. However, in general case of the BS equation we deal with the hyperradial excitations of two-dimensional surface, $\psi_L(p_0, |\mathbf{p}|)$.

3 Confinement for the Bethe-Salpeter equation

3.1 General discussion and non-relativistic confining potentials

The idea of confinement has different realization within different theoretical approaches. The simplest intuitive picture is given by the non-relativistic bound state formalism based on the Schrödinger equation with a QCD inspared phenomenological potential. A system of two particles interacting in a non-confining potential, vanishing as $r \to \infty$, has a spectrum of bound states with an upper limit, $M_{lim} = 2m_q$, and a continuum above this limit. The confinement is conventionally associated with the infinitely rising linear part of the full $q\bar{q}$-potential, $V_l = \alpha r, \alpha > 0$, which provides with the mass spectrum extending infinitely beyond M_{lim}. It is clear that this mechanism can not be directly adopted by the covariant field theoretical approaches, such as the BS formalism.

More relevant approach is based on the simultaneous analysis of the BS, Schwinger-Dyson (SD) equations and Ward-Takahashi at the lowest order and with the model gluon propagator [15, 14]. In particular, the role of the analytical structure (structure of singularities) of the quark propagators is discussed here. It is found that, with certain choices of the model gluon propagator, the quark propagator is an entire function (function with no singularities) at physical momentum of the quark. This important property of the quark propagator is considered as an indication of confinement [13, 14, 16].

These two examples present two essentially different pictures of what is referred to as confinement. In the first case, the confinement is the two-body effect, i.e. quarks can not escape from each other because of the interaction between them. In the second example, confinement is attributed to the property of a single quark, which can not propagate as a free particle.

The general formalism of the field theory suggests that the two-body and the single particle phenomena are in a generic relationship. So do the approaches based

on the BS and SD equations. Our approach to a modeling of the confinement is in some sense inverse to that of refs. [13, 14]. We, first, construct the covariant kernel of the BS equation which, we expect, would provide the confinement and only then study the modifications of the quark propagators involved in the BS equation.

We start from a few unsophisticated observations prompted by the non-relativistic picture:

1. In the non-relativistic limit the covariant theory can be reduced to the formalism with the Schrödinger equation, where we can expect the picture of the confinement as interaction with non-vanishing potential at $r \to \infty$ to be valid. The main distinguishing feature of the spectrum here is the existence of the bound states above the two quark mass limit, M_{lim}.

2. It is not necessary to have an infinitely rising potential, if we intend to discuss only a few lowest levels in the spectrum. More manageable potential, $V \to V_\infty > 0$ at $r \to \infty$, could be sufficient, if V_∞ is large enough. In this sense, we also refer to the constant potential as a confining one.

3. The potential in the momentum space $V(\mathbf{k})$ can be obtained as a non-relativistic limit of the kernel $\mathcal{K}$, similar to

$$V(\mathbf{k}) = -\frac{g^2}{\mathbf{k}^2 + \mu^2} \tag{25}$$

   if the kernel is of the form (2). Important point here is that the non-relativistic form of the potential (25) is of a field-theoretic origin.

Basing on the last observation we expect that if we define a way to construct a confining potential from the non-relativistic field theory, then we can apply similar methods to obtain the relativistic confining kernel. Very often the following recipe is used. The non-relativistic Yukawa potential in the coordinate space $V(r)$, the Fourier transform of eq. (25), is

$$V(\mathbf{r}) = -\frac{g^2}{4\pi}\frac{e^{-\mu r}}{r}. \tag{26}$$

The linear potential can be derived as

$$V_l = \lim_{\mu\to 0}\left[-\frac{\partial^2}{\partial\mu^2}V(\mathbf{r})\right] = \lim_{\mu\to 0}\frac{g^2}{4\pi}re^{-\mu r} = \frac{g^2}{4\pi}r. \tag{27}$$

The the relativistic generalization is made by a Fourier transform to the momentum space and replacing the non-relativistic Yukawa potential, (25), by it relativistic analog:

$$\mathcal{K} \propto \lim_{\mu\to 0}\left[-\frac{\partial^2}{\partial\mu^2}\frac{g^2}{k^2-\mu^2}\right]. \tag{28}$$

Taking the limit in (28), one should exercise great deal of care, since this leads to the appearance of generalized functions in the kernel [9, 11].

This recipe give us a guideline, however it is not completely satisfactory, since (i) the kernel (28) (or the potential (27)) is not of the field-theoretic form and (ii) it is not clear does the direct use of operation (28) lead to the rising or, at least, non-vanishing interaction in the four-dimensional space.

Intending to stay with our parametrization of the kernel as a superposition of the ladder terms, similar to (4), we have to find an appropriate presentation of the operation (27). Let us start with a superposition of the non-relativistic potentials:

$$V(\mathbf{r}) = \sum_j \frac{C_j}{r}\mathsf{exp}[-\mu_j r] = \sum_j \frac{C_j}{r}\mathsf{exp}[-\mu\alpha_j r], \tag{29}$$

where μ provides the mass scale and α_j are dimensionless parameters. Then expanding the exponents we get

$$V(\mathbf{r}) \;=\; \sum_j C_j\left[\frac{1}{r} - \mu\alpha_j + \mu^2\frac{\alpha_j^2 r}{2} - \ldots\right]. \tag{30}$$

From eq. (30) we see that, taking the limit $\mu \to 0$ and correspondly adjusting the parameters C_j and α_j, desired non-vanishing behavior of the potential can be provided. For instance, for constant potential, V_c, we have

$$C_j = \mu^{-1}\tilde{C}_j; \quad \sum_j \tilde{C}_j = 0; \quad \sum_j \alpha_j\tilde{C}_j \equiv A_c \neq 0, \tag{31}$$

where we need only two terms to satisfy the conditions, i.e. $j_{max} = 2$. For linear potential,V_c,

$$C_j = \mu^{-2}\tilde{C}_j; \quad \sum_j \tilde{C}_j = 0; \quad \sum_j \alpha_j \tilde{C}_j = 0; \quad \sum_j \alpha_j^2 \tilde{C}_j \equiv A_l \neq 0, \tag{32}$$

where $j_{max} = 3$. The limit $\mu \to 0$ corresponds to the physical picture of the superposition of very light mass exchanges with slightly different masses and large coupling constants. Please, note that power of the non-vanishing term in the limit $\mu \to 0$ is solely controlled by the power of the μ in the denominator of the coefficients C_j.

Using a superposition of the Yukawa potentials in the momentum space, we find (see Appendix A):

$$V_c(\mathbf{k}) = A_c \delta^{(3)}(\mathbf{k}), \tag{33}$$

$$V_l(\mathbf{k}) = \left(\frac{1}{2} - \frac{\mathsf{ln}2}{\pi}\right) A_l \delta^{(3)}(\mathbf{k}) \frac{\partial}{\partial k}. \tag{34}$$

Note that, for the linear potential, the Schrödinger equation becomes an integro-differential equation in momentum space.

3.2 Confining kernel for the Bethe-Salpeter equation

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We look for a confining kernel, $\mathcal{K}_{con}$, of the BS equation in the form of the superposition of ladder kernels in the momentum space and after the Wick rotation:

$$\mathcal{K}_{con}(k_E) = \sum_j \frac{C_j}{k_E^2 + \alpha_j^2 \mu^2}, \tag{35}$$

where $k_E^2 = k_0^2 + \mathbf{k}^2$.

In spite of the obvious similarity to the non-relativistic case, the expression for the relativistic kernel, (35), has essentially different properties in view of the larger dimension of the space. Indeed, the ladder kernel in the coordinate space is [6]:

$$\mathcal{K}_{ladder}(R_E, \mu) = g^2 \mu \frac{\mathsf{K}_1(\mu R_E)}{R_E}, \tag{36}$$

where $R_E = (\mathbf{r}^2 + t^2)^{1/2}$. The asymptote at small R_E can be obtained as:

$$\mathcal{K}_{ladder}(R_E \to 0, \mu) \sim g^2 \left\{ \frac{1}{R_E^2} \right.$$
$$\left. + \frac{\mu^2}{2} \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!} \left(\frac{\mu R_E}{2}\right)^m \left[\mathsf{ln}\left(\frac{\mu R_E}{2}\right) - \frac{1}{2}\left(\psi(m+1) + \psi(m+2)\right)\right]\right\}, \quad (37)$$

where $\psi(m)$ is the Euler's psi function. The expansion (37) is different from the behavior of the non-relativistic potentials. However, a procedure similar to that of the non-relativistic case can be applied to cancel the lowest order terms in the expansion (37) and generate a non-vanishing kernel in the limit $\mu \to 0$, since at large R_E there is the exponential suppression similar to the one in the Yukawa potential:

$$\mathcal{K}_{ladder}(R_E \to \infty, \mu) \sim g^2 \mu^{1/2} \frac{\mathsf{exp}[-\mu R_E]}{R_E^{3/2}}. \quad (38)$$

We can see that the direct use of operation (28) does not lead to the desired non-vanishing at $R_E \to \infty$ behavior of the kernel.

We study here the kernel with the lowest power in μ^{-1}, which as we expect controls the asymptotic behavior at large R_E. Analysis in the momentum space, similar to that of the non-relativistic case, gives the following conditions for the coefficients C_j (see Appendix B):

$$C_j = \mu^{-2}\tilde{C}_j; \quad \sum_j \tilde{C}_j = 0; \quad \sum_j \alpha_j^2 \tilde{C}_j = 0; \quad \sum_j \alpha_j^2 \mathsf{ln}\alpha_j \tilde{C}_j \equiv A \neq 0, \quad (39)$$

which can be satisfied explicitly for $j_{max} = 3$. Note, that the second condition cancels the most singular terms, $\sim R_E^{-2}$, in the coordinate space, (37), similar to the non-relativistic potentials. This choice of conditions leads to a kernel of the form:

$$\mathcal{K}_{con}(k) = -(2\pi)^4 U^4 \delta^{(4)}(k_E), \quad (40)$$

where for simplicity we introduce new effective coupling constant U, which has dimension of mass, and sign is chosen to provide us with a confining-like kernel, similar to the positive constant potential in the non-relativistic case. Fourier transform of the kernel, (40), is a constant, $\sim U^4$, in four dimension. This means that it does

not behave like a non-relativistic constant potential, for which we expect behavior like $\sim \delta(t_0) \cdot constant$. Therefore, the kernel (40) does not exactly correspond to the non-relativistic constant potential and the effective constant, U^4, is not related to constant in such a potential. Note, since the constant U^4 does not have the direct physical meaning, the choice of the factor $(2\pi)^4$ is arbitrary and it is made for further simplification of formulae.

The form of the kernel, (40), in accordance with our main idea, is considered as a special limiting case of the sum of the ladder kernels (sum of one-boson exchanges), which provides explicit covariance of the kernel and connection with the usual field-theoretic constructions. (Note that this is a valid form in the Euclidian space, whereas the transition to the Minkowsky space is not defined.) By itself the δ-form of the kernel is not something very unusual in studying of the bound states of the quarks. For instance, such a form is considered as "regularized" form of the highly singular, $\sim k^{-4}$, behavior of the gluon propagator [13] and is a basis of the models for studying of the SD equation for the quark propagator [15, 14]. In particular, this form of the gluon propagator leads to the quark propagator without singularities along physical momentum. In the lowest order, such a gluon propagator gives the kernel of the BS equation [15].

Let us study the effect of the kernel, (40), in the BS equation, (1), under the Wick rotation. The form of $\mathcal{K}_{con}$ allows an integration in the equation explicitly:

$$\psi(p,P) = -\frac{U^4}{D(p_1,p_2)}\psi(p,P). \tag{41}$$

From eq.(41) we find that the kernel, $\mathcal{K}_{con}$, does not allow for the bound state solutions of the BS equation. In this sense we can expect that this kernel, in effect, is similar to the constant non-relativistic potential, which along does not allows for the bound states. If this is the case, we expect that adding this kernel to the regular attractive kernel, i.e. like the one presented in Section 2.3, Table 1, we will get a shift of the spectrum by a constant.

The BS equation with the combined kernel, $\mathcal{K}_G + \mathcal{K}_{con}$, can be transformed to

the usual form, but with a modified two-body propagator:

$$\psi(p,P) = \frac{1}{D(p_1,p_2)+U^4}\int \frac{d^4p'}{(2\pi)^4}\mathcal{K}_G(p,p',P)\psi(p',P). \qquad (42)$$

New propagator in eq. (42) has different analytical properties, compared to the initial free propagator under the Wick rotation, (6), and main difference is that the new propagator does not have singularities at $M > 2m$, which were a signal of the limit of the spectrum at this point. This is an indication that the *physical spectrum* exists beyond this point. However, without numerical analysis we are not able to discuss the properties of the solutions at $M > 2m$.

Before to go to the numerics, let us discuss the possible effects of the self energy corrections in the presence of the interaction of the form (40). Dealing with the kernel (40) as a sum of the lowest order (ladder) kernels, we calculate the one-loop self-energy corrections to the single quark propagator, $d(p)^{-1}$. Integration over the loop is performed at imaginary p_0 component of the four-momentum of the quark and result can be analytically continued to any values of p_0. For physical momentum we get:

$$d(p) = p^2 - m^2 + \frac{U^4}{p^2-m^2}. \qquad (43)$$

The propagator (43) does not have singularities for physical values of the momentum, p. This property of the interaction of the form (40) has already been established within model investigations of the behavior of the quark propagator [14, 13]. Now we see that this single quark effect is in a generic relation with the two-body confining interaction in the framework of the BS equation. For our calculations it is also important that the singularities of the modified two-body propagator in (42) with $D(p_1,p_2) = d(p_1)\cdot d(p_2)$ defined by eq. (43) still allow to perform the Wick rotation.

3.3 Numerical investigation of the Bethe-Salpeter equation with the confining kernel

First we compare the meson spectra obtained with eq. (42) with $U^4 = 0$ and $U^4 \neq 0$. For convenience we take the same kernel, $\mathcal{K}_G$, as that in the section 2.3. The

main effect, we expect here, is the shift of the spectrum beyond the limiting point $M_{lim} = 2m$. In order to compare the spectra, we calculate $\lambda(M)$ for two lowest states in the channels with $L = 0, 1$. This is enough to make conclusions about (i) the position of the bound states, (ii) limiting point of the spectra and (iii) separation between ground and excited states. Results of these calculations are presented in the Fig. 3, groups of curves A and B. The constant U for the case B is chosen to be of the typical energy scale of the equation, $U = m$.

We find, indeed, that the masen spectrum of the equation with $U \neq 0$ is extended beyond the point M_{lim}. However, it displays unusual behavior beyond the point, M^B_{lim} (this point is shown on Fig. 3 by the arrow) [23]. Non-monotonical behavior of the curves $\lambda(M)$ indicates some difficulties in the interpretation of the corresponding solutions. The obvious difficulty is the existence of two solutions with the same coupling constant λ and different masses, M, (see e.g. discussions in refs. [19]). Therefore, we have to find a way to isolate the only one physical solutions. The solution of this problem is quite simple. Calculating the normalization of the solutions of the equation along M we find that the ground state solutions beyond M^B_{lim} have a negative norm, i.e. are the *abnormal* non-physical solutions. [24]. This observation gives us the selection rule to eliminate the extra solutions.

Another problem with the solutions corresponding $U^4 \neq 0$ is that spectrum is not only shifted to larger masses, but also the separation between levels is drastically increased (see Table II). This effect make it a problem for a phenomenological use of the kernel of the form (40). Indeed, if we intend to consider states above M_{lim} we have to take U^4 large enough to provide us with a new limit of the spectrum, however this can be in conflict with the desired separation between levels. One may try to adjust the parameters of the remaining part of the full kernel, $\mathcal{K}_G$, so as to have a reasonable density of levels. However our analysis showed that in the presence of the kernel, $\mathcal{K}_{con}$, separation between levels depends only weakly on the kernel, $\mathcal{K}_G$.

Δ M	A $\lambda^A_{phys} = 0.13$	B $\lambda^B_{phys} = 0.25$	C $\lambda^C_{phys} = 0.44$	D $\lambda^D_{phys} = 1.44$
1P - 1S	0.49 GeV	0.77 GeV	0.48 GeV	0.48 GeV
2S - 1P	0.19 GeV	0.47 GeV	0.16 GeV	0.17 GeV
2P - 2S	0.21 GeV	0.42 GeV	0.22 GeV	0.26 GeV

Table. II. The levels splitting in the "scalar charmonium" for different kernels .

It should be remembered that in the BS equation the addition of interaction of any kind to the full kernel cannot be related linearly to the shift in the mass of the system. The same interaction also "shifts" the masses of the constituents through the self-energy corrections to the single-particle propagators. A self-consitent approach has to be adopted to include both the self-energy corrections and changes in the two-body interactions.

We take into account the self-energy correction, (43), to the quark propagator. This leads to a corresponding modification of the two-body propagator $D(p,P)^{-1}$, (3). The resulting spectrum is presented in Fig. 3, group of curves C. It is clear that for the interval of masses $\sim 2 \div 3.5$ GeV the density of levels "returns to normal", of the same value as in the case A (see Table II). The value of constant $\lambda^C_{phys} = 0.44$ is chosen as an example giving the spectrum density close to the the one of the case A. At smaller λ^C_{phys} the separation of levels is even smaller.

The examples of calculations, B and C, show that the kernel containing the part $\mathcal{K}_{con}$ is indeed similar in its effect to the non-relativistic potential in coordinate space, $V(r)$, with $V(r \to \infty) \to V_\infty$. where the positive real constant V_∞ defines the shift of the bound state spectrum compared to the case with no $\mathcal{K}_{con}$, the case A. From Fig. 3 another similarity with the non-relativistic constant potential is obvious, this kernel gives only a limited number of states in the spectrum, which can be adjusted by varying the λ_{phys}. For instance, for the case C with $\lambda^C_{phys} = 0.2$ there is only one bound state, $1S$, whereas with $\lambda^C_{phys} = 0.45$ there are four states in the S and P channels (there can be other undetected states in channels with higher $L = 2, \ldots$).

On the other hand, taking account of the self-energy corrections, corresponding

to this type of interaction, leads to the disappearance of the poles in the single quark propagator. Therefore quarks cannot propagate other than being bound in a bound state. Apparently, this fact is not related to the infinitely rising interaction between them, but rather to the modification of the single quark properties.

To be sure that the picture is valid in a wide interval of the constant U^4, also the spectrum is calculated for the case $U^4 = 4m^4$ and self-energy corrections taken into account. The result is shown in the Fig. 3, group of curves D. That the picture is found to be similar to the one of case C, but the spectrum is shifted up to even larger $M^D_{lim} \sim 4$ GeV. The density of levels in the interval of masses $\sim 2.5 \div 4$ GeV is the same as those of cases A and C (see Table II). The value of constant $\lambda^C_{phys} = 0.44$ is chosen again to give the spectrum density close to the the one of the case A.

Note that in our model the constant U can not be taken arbitrary large, since all the calculations are performed in the lowest order. The natural criteria on the maximum value of U is that the corrections of the lowest order, $\sim U^4$, must not be too large. For instance, a shift of the mass spectrum with $U \neq 0$ should not be too big compare to the typical masses for the case with $U = 0$.

4 Conclusions

We have presented a method to find the ground state and excited states of the two-body Bethe-Salpeter equation for a channel with any quantum numbers. This method allows us to solve the bound state problem without reduction of equation to quasipotential form or any other approximations.

Based on a qualitative analogy with the construction of a non-relativistic potential with non-vanishing asymptote at large distances, $r \to \infty$, we have proposed a recipe to obtain the confining kernel for the Bethe-Salpeter equation, parametrized in the form of a special limiting case of a superposition of ladder kernels. We find that in the simplest case such a kernel is proportional to $\delta(k_E)$ in the Euclidean momentum space, which corresponds to a constant kernel in the coordinate space.

We have studied the effect of this kernel on the spectrum of the Bethe-Salpeter

equation, when the usual attractive interaction is added. It is found that this kernel is similar in its effect to the non-relativistic potential in coordinate space, $V(r)$, with $V(r \to \infty) \to V_\infty$. The positive real constant V_∞ gives the scale that defines the limit of the bound state spectrum compared to the sum of the constituent masses, $M < 2m + V_\infty$. At the same time the self-energy corrections remove the singularities from the propagators of the constituents, i.e. constituents do not propagate. Combination of these features of the solutions allows an interpretation of this type of interaction as a confining interaction.

The illustrative analytical and numerical calculation are presented for a model of massive scalar particles with scalar interaction and do not pretend to a phenomenological application. However, the developed formalism can be straightforwarly adopted for the Bethe-Salpeter equation for the bound state of two spinor fields and, therefore, can be used for the realistic studies of the properties of the quark-antiquark systems, mesons.

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Appendix A. The non-relativistic confining potentials in the momentum space.

In order to establish the form of the potentials defining by the eq. (29), (31) and (32), let us consider auxiliary integral, I_a:

$$I_a = \int \frac{d^3\mathbf{k}}{(2\pi)^3} V_a(k) f(\mathbf{k}) \tag{A.1}$$

$$= \int \frac{d\Omega}{(2\pi)^3} \int_0^\infty dk k^2 V_a(k) f(k, \Omega), \tag{A.2}$$

where $f(\mathbf{k})$ is any function for which we assume a "good" behavior ($f \to 0$ when $k \to \infty$, no singularities, existence of derivatives, etc.), $a = c, l$ depending on which of conditions the (31) or (32) is imposed on the potential and $V_a(k)$ defined as a fourier transform of eq. (29):

$$V_a(k) \;=\; 4\pi \sum_j \frac{4\pi C_j}{k^2 + \alpha_j^2 \mu^2}. \tag{A.3}$$

The limit $\mu \to 0$ is assumed and we take it later.

Using the common condition $\sum C_j = 0$, we rewrite eq. (A.2) as

$$I_a \;=\; -4\pi\mu^2 \int \frac{d\Omega}{(2\pi)^3} \sum_j C_j \alpha_j^2 \int_0^\infty \frac{dk}{k^2 + \alpha_j^2 \mu^2} f(k, \Omega). \tag{A.4}$$

Integrating by parts the last integral in (A.4), we get:

$$I_a \;=\; 4\pi\mu \int \frac{d\Omega}{(2\pi)^3} \sum_j C_j \alpha_j \int_0^\infty dk \;\mathsf{arctg}\left[\frac{k}{\alpha_j \mu}\right] f'(k, \Omega), \tag{A.5}$$

where $f' = \partial f / \partial k$.

Let us now consider integration over k only. These integrals on the r.h.s. of eq. (A.5) can be split in two parts:

$$I_a \propto \mu \sum_j C_j \alpha_j \left\{ \int_0^{\alpha_j \mu} + \int_{\alpha_j \mu}^\infty \right\} dk \;\mathsf{arctg}\left[\frac{k}{\alpha_j \mu}\right] f'(k, \Omega). \tag{A.6}$$

We estimate the first integral by the mean value theorem:

$$\mu \sum_j C_j \alpha_j \int_0^{\alpha_j \mu} dk \ldots = \mu^2 \left(\frac{\pi}{4} - \frac{\mathsf{ln}2}{2}\right) \sum_j C_j \alpha_j^2 f'(\xi \alpha_j \mu, \Omega), \tag{A.7}$$

where $0 \le \xi \le 1$.

The second integral is estimated using the expansion $\mathsf{arctg}x = \pi/2 - 1/x + 1/(3x^3) - \ldots$, which is valid at $x \ge 1$. It can be shown that the first term in this expansion gives the leading contribution to the full integral in the limit $\mu \to 0$:

$$\mu \sum_j C_j \alpha_j \int_{\alpha_j \mu}^\infty dk \ldots = -\frac{\pi}{2} \mu \sum_j C_j \alpha_j f(\alpha_j \mu, \Omega). \tag{A.8}$$

Finally, taking the limit $\mu \to 0$ and accounting for the conditions on the coefficients C_j, we find

$$I_c = A_c f(\mathbf{0}), \tag{A.9}$$

$$I_l = \left(\frac{1}{2} - \frac{\ln 2}{\pi}\right) A_l f'(\mathbf{0}). \tag{A.10}$$

These equations give us the potentials in the form (33) and (34).

Appendix B. The relativistic confining kernel in the momentum space.

In order to establish the form of the kernel defined by eq. (35) at the limit $\mu \to 0$ and lowest (but non-zero) degree of μ^{-1} in the C_j, let us consider the auxiliary integral, I:

$$I = \int \frac{d^4\underline{k}}{(2\pi)^4} \mathcal{K}_{con}(k) f(\underline{k}) \tag{B.1}$$

$$= \int \frac{d\Omega^{(4)}}{(2\pi)^4} \int_0^\infty dk k^3 \mathcal{K}_{con}(k) f(k, \Omega^{(4)}), \tag{B.2}$$

where $\underline{k}$ is four-momentum in Euclidian space, $k = (\mathbf{k}^2 + k_0^2)^{1/2}$, $\Omega^{(4)}$ is the hyperangle defining the orientation of the vector $\underline{k}$ in the four dimensional space; $f(k)$ is an arbitrary function for which we assume "good" behavior ($f \to 0$ when $k \to \infty$, no singularities, existence of derivatives, etc.). The limit $\mu \to 0$ is assumed and we take it later. We are omitting all nonessential factors, such as 2π, etc., in the following calculations.

Adding and subtracting expression C_j/k^2 to each term in (35), we rewrite eq. (B.2) as

$$I \propto \quad -\mu^2 \int d\Omega^{(4)} \sum_j C_j \alpha_j^2 \int_0^\infty \frac{dk\, k}{k^2 + \alpha_j^2 \mu^2} f(k, \Omega^{(4)}) \tag{B.3}$$

$$+ \int d\Omega^{(4)} \sum_j C_j \int_0^\infty \frac{dk}{k^2} f(k, \Omega^{(4)}).$$

Th last term is cancelled by the condition

$$\sum_j C_j = 0. \tag{B.4}$$

Let us now consider integration over k only. This integral in r.h.s. of eq. (B.3) can be evaluated as:

$$I \propto -\mu^2 \sum_j C_j \alpha_j^2 f(\alpha_j \mu, \Omega^{(4)}) \int_0^\Lambda \frac{dk\ k}{k^2 + \alpha_j^2 \mu^2}, \tag{B.5}$$

since the function remaining under integration has sharp maximum at $k = \alpha_j \mu$. We also introduce the cut-off parameter Λ to regularize formally the logarithmically divergent integrals. At a later stage of calculation we take the limit $\Lambda \to \infty$, but for the moment it is enough to assume $\Lambda \gg \mu$. Performing the integration we get

$$I \propto -\mu^2 \sum_j C_j \alpha_j^2 f(\alpha_j \mu, \Omega^{(4)}) \left[\mathsf{ln}\left(\frac{\Lambda}{\mu}\right) - \mathsf{ln}\alpha_j \right]. \tag{B.6}$$

Applying the condition

$$\sum_j C_j \alpha_j^2 = 0, \quad \sum_j C_j \alpha_j^2 \mathsf{ln}\alpha_j \neq 0, \tag{B.7}$$

and taking the limit $\mu \to 0$, we get

$$I \propto \mu^2 f(0) \sum_j C_j \alpha_j^2 \mathsf{ln}\alpha_j, \tag{B.8}$$

which proves the form of the kernel as eq. (40).

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[24] The same is valid for all trajectories of the group B beyond their minima.

Figure captions

Figure 1. Mapping of the bound state spectrum of the Bethe-Salpeter equation. The physical states correspond to the cutting of the trajectories $\lambda(M)$ by the line $\lambda = \lambda_{phys}$. The three lowest trajectories for $L = 0$ (solid curves) and $L = 1$ (dashed curves) are presented for the parameters of the kernel from Table I. An example of physical cut is shown for $\lambda_{phys} = 0.13$ and corresponding masses are presented in (23)-(24).

Figure 2. The amplitudes corresponding to the three lowest $S-$states ($L = 0$) for the Bethe-Salpeter equation with interaction from the Table I. These amplitudes corresponding to the states with masses eq. (23) and $\lambda_{phys} = 0.13$. Curves: ground state ($1S$) - solid; first excited state ($2S$) - dashed; second state ($2S$) - dotted.

Figure 3. Mapping of the bound state spectrum of the Bethe-Salpeter equation with different kernels. The group of curves, A, is the two lowest state from the Fig. 1 (see caption). Solid curves present the S-states and the dashed curves presents the P-states. The groups B, C and D are trajectories for the same kernel as the group A plus the confining part, $\mathcal{K}_{con}$: B and C with $U^4 = m^4$; D with $U^4 = 4m^4$. The calculations of groups C and D also take into account the self-energy corrections. Arrows along the $M-$axis show the limits of the corresponding physical spectra.